Contemporary Mathematics

# Rationality of varieties over fields

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Abstract. We survey rationality results for algebraic varieties in dimensions up to three, with a focus on recent developments for geometrically rational threefolds over non-closed fields.

## 1. Introduction

Let $X$ be a smooth projective variety over a field $k$. There are several notions measuring how "close" $X$ is to projective space, including:

- $X$ is $k$-rational if $X$ is birationally equivalent to projective space $\mathbb{P}_k^{\dim X}$;
- $X$ is stably $k$-rational if $X \times \mathbb{P}_k^r$ is $k$-rational for some integer $r$;
- $X$ is $k$-unirational if there exists a dominant rational map $\mathbb{P}_k^{\dim X} \dashrightarrow X$;
- $X$ is $k$-rationally connected if for every algebraically closed field extension $L/k$, any two general $L$-points can be joined by a rational curve over $L$.

We will sometimes omit the reference to the ground field $k$ if it is clear from context. The following implications hold:

$$k\text{-rational} \implies \text{stably } k\text{-rational} \implies k\text{-unirational} \implies k\text{-rationally connected.}$$

If $X$ is $k$-rational, then the base change $X_L$ is $L$-rational for any field extension $L/k$. In particular, if $X$ is $k$-rational, then it is necessarily geometrically rational over $k$ (meaning that $X_{\overline{k}}$ is $\overline{k}$-rational, where $\overline{k}$ is the algebraic closure).

Question 1.1. Let $X$ be a smooth projective geometrically rational variety over a field $k$. When is $X$ $k$-rational?

This survey article aims to give an overview of currently known results about Question 1.1, especially focusing on recent developments in dimension three. There are many excellent surveys about rationality of algebraic varieties, which discuss many aspects of the rationality problem not covered here; see, e.g., [**AB17, Bea16, BT18, CT19, CTS07, Deb24, KSC04, Pir18, Sch25, Voi16**]. Here, we will focus on recent developments for varieties over non-closed fields, primarily in dimensions up to three. We will not give any proofs or detailed computations. Instead, we aim to give an overview of some results and methods, and we will provide references for proofs and additional details.

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**Notation.** Throughout $k$ will denote a field and $\overline{k}$ will denote a fixed algebraic closure of $k$. Let $k^{\mathrm{perf}}$ be the perfect closure of $k$ in $\overline{k}$, and let $G_k = \mathrm{Gal}(\overline{k}/k^{\mathrm{perf}}) = \mathrm{Aut}(\overline{k}/k)$ denote the absolute Galois group of $k$. A variety over $k$ is a separated integral scheme of finite type over $k$. If $X$ is a smooth projective variety over $k$, then $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on $X$ defined over $k$, $\mathbf{Pic}_{X/k}$ is its Picard scheme, and $\rho(X)$ is the Picard rank of $X$ (over $k$). Note that $\mathrm{Pic}(X) \subset \mathbf{Pic}_{X/k}(k) = \mathrm{Pic}(X_{\overline{k}})^{G_k} \subset \mathrm{Pic}(X_{\overline{k}})$ and $\rho(X) \leq \rho(X_{\overline{k}})$.

## 2. Some rationality constructions and obstructions

There are two directions in the rationality problem. In one direction, in order to show that $X$ is rational, one needs a rationality construction, i.e., a birational map from $\phi\colon X \dashrightarrow \mathbb{P}_k^{\dim X}$. In the other direction, in order to show that $X$ is not rational, one must show that no such map $\phi$ can exist. This can sometimes be accomplished by finding a rationality obstruction, i.e., by finding a birational invariant that differs for $X$ and projective space. In this section, we give a few examples of rationality constructions and obstructions. First, here are some examples of rationality constructions.

Example 2.1. Let $Q \subset \mathbb{P}_k^{n+1}$ be a smooth quadric, and assume $Q(k) \neq \emptyset$. For each $p \in Q(k)$, the projection $\pi_p\colon Q \dashrightarrow \mathbb{P}^n$ from $p$ is defined over $k$ and is generically one-to-one, so it exhibits a birational equivalence between $Q$ and $\mathbb{P}_k^n$.

Example 2.2. Let $\pi\colon X \to \mathbb{P}_k^m$ be a quadric fibration with smooth generic fiber. Assume $\pi$ has a section (for example, if $m = 1$ and $k$ is algebraically closed, this assumption holds by Tsen's theorem). Then the generic fiber of $\pi$ is a smooth quadric with a rational point over the function field $k(\mathbb{P}^m)$, so it is $k(\mathbb{P}^m)$-rational by Example 2.1. This implies that the function field of $X$ is purely transcendental over $k$; that is, $X$ is $k$-rational.

Example 2.3 (See [**CTSSD87**, Proposition 2.2]). Let $n \geq 2$, let $k$ be a field of characteristic $\neq 2$, and let $X \subset \mathbb{P}_k^{n+2}$ be a smooth complete intersection of two quadrics. Assume $X$ contains a line $\ell$ defined over $k$ (for example, this always holds if $k$ is algebraically closed, see [**CTSSD87**, Remark 3.3.1]). Then the projection $\pi_\ell\colon X \dashrightarrow \mathbb{P}_k^n$ from $\ell$ is a birational map, so $X$ is $k$-rational.

To see this explicitly, choose coordinates on $\mathbb{P}_k^{n+2}$ so that $\ell$ is defined by $(x_2 = \cdots = x_{n+2} = 0)$. Then $X$ is defined by two quadratic equations

$$\phi_i = x_0 l_{i0} + x_1 l_{i1} + q_i \quad \text{for } i \in \{0, 1\},$$

where $l_{ij}, q_i \in k[x_2, \ldots, x_{n+2}]$ are homogeneous polynomials with $\deg(l_{ij}) = 1$ and $\deg(q_i) = 2$. The singular members of the pencil spanned by $\phi_1$ and $\phi_2$ are cones over smooth quadrics in $\mathbb{P}^{n+1}_k$ [**CTSSD87**, Lemma 1.13], so the matrix

$$\begin{pmatrix} l_{00} & l_{01} \\ l_{10} & l_{11} \end{pmatrix}$$

is invertible at a general geometric point of $X$. On the open subset $(x_2 \neq 0)$ of $\mathbb{P}^{n+2}_k$, $X$ is defined by

$$\begin{cases} \frac{x_0}{x_2} l_{00}(1, \frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2}) + \frac{x_1}{x_2} l_{01}(1, \frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2}) = -q_0(1, \frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2}) \\ \frac{x_0}{x_2} l_{10}(1, \frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2}) + \frac{x_1}{x_2} l_{11}(1, \frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2}) = -q_1(1, \frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2}) \end{cases}$$

so we see that on $(x_2 \neq 0)$, $\pi_\ell$ defines a birational map from $X$ to affine space $\mathbb{A}^n_k$ with coordinates $(\frac{x_3}{x_2}, \dots, \frac{x_{n+2}}{x_2})$.

Here is another perspective on Example 2.3 over an algebraically closed field.

Example 2.4 ([**CTSSD87**, Theorem 3.2]). Let $n \geq 2$, let $X \subset \mathbb{P}^{n+2}_k$ be a smooth complete intersection of two quadrics, and assume $k$ is an algebraically closed field of characteristic $\neq 2$. After a change of coordinates on $\mathbb{P}^{n+2}_k$, we may assume that $[1 : 0 : \cdots : 0]$ is a point of $X$, so that $X$ is defined by two quadratic equations of the form $\phi_i = x_0 l_i + q_i$ for $i \in \{0, 1\}$, where $l_i, q_i \in k[x_1, \dots, x_{n+2}]$ are homogeneous polynomials of the correct degrees. Consider the variety $Z \subset \mathbb{P}^1_k \times \mathbb{P}^{n+1}_k$ defined by

$$sl_0 + tl_1 = sq_0 + tq_1 = 0,$$

where $[s : t]$ are coordinates on $\mathbb{P}^1_k$. Since

$$[x_0 : x_1 : \cdots : x_{n+2}] \mapsto ([-l_1(x_1, \dots, x_{n+2}) : l_0(x_1, \dots, x_{n+2})], [x_1 : \cdots : x_{n+2}])$$

defines a rational map $X \dashrightarrow Z$ that has an inverse on $Z \setminus (l_0 = l_1 = 0)$, we see that $Z$ is birationally equivalent to $X$.

The first projection $\pi_1 \colon Z \to \mathbb{P}^1_k$ is a fibration in $(n-1)$-dimensional quadrics. This quadric fibration has a section by Tsen's theorem, so $Z$ is rational by Example 2.2, and hence $X$ is rational.

Now we give a few examples of birational invariants that can be used to obstruct rationality. First, the geometric genus $p_g(X) = h^0(X, \mathcal{O}_X(K_X))$ is a birational invariant of smooth projective varieties. In fact, the proof given in [**Har77**, Theorem II.8.19] can be adapted to show the following statement about plurigenera.

Proposition 2.5 (Proof of [**Har77**, Theorem II.8.19]). *If $Y \dashrightarrow X$ is a separable dominant rational map of smooth projective $n$-dimensional varieties, then $h^0(Y, \mathcal{O}_Y(mK_Y)) \geq h^0(X, \mathcal{O}_X(mK_X))$ for all $m \geq 0$.*

In particular, if $X$ is a smooth projective separably $k$-unirational variety (i.e., if there is a separable dominant rational map $\mathbb{P}^n_k \dashrightarrow X$), then $h^0(X, \mathcal{O}_X(mK_X)) = 0$ for all $m \geq 1$. This shows that smooth projective curves of genus $\geq 1$ are not (separably uni)rational.

Example 2.6. Let $X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree $d$. If $d \geq n+2$, then $K_{X_d} = \mathcal{O}_{X_d}(d-n-2)$ is ample or trivial, so $X_d$ is not unirational by Proposition 2.5 (and in fact is not rationally connected). On the other hand, if $d \leq n+1$, then $X_d$ is Fano and therefore is rationally connected by [**Cam92, KMM92**]. If $d \leq 2$, then $X_d$ is rational by Example 2.1; however, in general, the rationality problem for Fano hypersurfaces is an old and difficult problem. We refer the reader to [**Sch25**] for a survey on currently known results.

EXAMPLE 2.7. The birational automorphism group $\mathrm{Bir}(X)$ of a $k$-variety $X$ is the group of birational self-maps $X \dashrightarrow X$ defined over $k$. If $X$ and $Y$ are two birationally equivalent $k$-varieties, then their birational automorphism groups must be isomorphic. To give a concrete example of an application, this gives another way to see that a complex curve $C$ of genus $\geq 2$ is not rational, since $\mathrm{Aut}(C) = \mathrm{Bir}(C)$ is finite but $\mathrm{Bir}(\mathbb{P}^1_{\mathbb{C}}) = \mathrm{PGL}_2(\mathbb{C})$ is infinite.

In fact, a remarkable result of Cantat and Regeta–Urech–van Santen shows that a complex variety $X$ is birationally equivalent to $\mathbb{P}^n_{\mathbb{C}}$ if and only if $\mathrm{Bir}(X)$ and $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ are isomorphic as groups [**Can14, RUvS25**].

The previous two rationality obstructions have primarily been used over $\mathbb{C}$. The next obstruction is specific to non-closed fields $k$, and it can sometimes detect non-rationality of geometrically rational varieties.

PROPOSITION 2.8 (Lang–Nishimura. See, e.g., [**RY00**, Proposition A.6]). *Let $Y \dashrightarrow Y'$ be a rational map of varieties over a field $k$. If $Y$ has a smooth $k$-point and $Y'$ is projective, then $Y'(k) \neq \emptyset$.*

In particular, a smooth projective $k$-(uni)rational variety necessarily has a $k$-point. Both the smoothness and projectivity assumptions are necessary here: for a singular counterexample, take $Y$ to be the cone over a smooth projective variety with no $k$-points and $Y'$ to be the blow-up of the vertex, and for a non-projective counterexample, consider $Y' = \mathbb{P}^n \setminus \mathbb{P}^n(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$.

By Example 2.1 and Proposition 2.8, a smooth quadric $Q \subset \mathbb{P}^{n+1}_k$ is $k$-rational if and only if it is $k$-unirational if and only if it has a $k$-point. This shows that Question 1.1 has a negative answer in general, as there exist smooth quadrics without rational points. Concretely, over the real numbers, one may take the quadric in $\mathbb{P}^{n+1}_{\mathbb{R}}$ defined by the positive definite quadratic form $\sum_{i=0}^{n+1} x_i^2$.

Over the real numbers, if $X$ is a smooth projective variety of dimension $n$, then the set $X(\mathbb{R})$ of its real points endowed with the Euclidean topology is a disjoint union of compact connected smooth manifolds of dimension $n$. This yields additional obstructions to rationality. If $X$ is $\mathbb{R}$-rational, then $X(\mathbb{R})$ is nonempty by Proposition 2.8. The following obstruction involving the number of connected components of $X(\mathbb{R})$ dates back to work of Comessatti [**Com12**]:

THEOREM 2.9 (See [**CT19**, Théorème 3.32]). *The number of connected components of $Y(\mathbb{R})$ is a stable birational invariant of smooth projective geometrically connected varieties $Y$ over $\mathbb{R}$.*

In particular, if $X$ is a smooth projective (stably) $\mathbb{R}$-rational variety, then $X(\mathbb{R})$ is nonempty and connected. Both the smoothness and projectivity assumptions are necessary in Theorem 2.9. For a singular counterexample, consider the nodal plane curve defined by $y^2 - x^2(x-2)$ [**Man20**, Example 2.2.13]. This curve is $\mathbb{R}$-rational, as its normalization is $\mathbb{P}^1_{\mathbb{R}}$. However, the real locus of this curve is disconnected: it consists of the disjoint union of the singular point and a one-dimensional component.

## 3. Curves and surfaces

In this section, we consider curves and surfaces. As we shall see, the rationality question has a complete answer in dimensions one and two. Over $\mathbb{C}$, the notions of rationality, stable rationality, unirationality, and rational connectedness all coincide for curves and surfaces. Over an arbitrary field $k$, to determine the $k$-rationality of a geometrically rational curve or surface, one must additionally consider the existence of $k$-points, and, in dimension two, additional data coming from how the absolute Galois group $G_k$ acts on the geometric Picard group $\mathrm{Pic}(S_{\overline{k}})$ of the surface $S$.

**3.1. Curves.** Let $C$ be a smooth projective curve over $k$. Since $\mathbb{P}^1_k$ has genus zero, by Proposition 2.5 we see that if $C$ is $k$-rational then it necessarily has genus zero. On the other hand, if $C$ has genus zero, then the anticanonical divisor satisfies $\deg(-K_C) = 2$ and hence is very ample. The linear system $|-K_C|$ has dimension 2 by Riemann–Roch, and therefore it defines an embedding of $C$ into $\mathbb{P}^2_k$ as a conic. Then Example 2.1 and Proposition 2.8 show that $C$ is $k$-rational if and only if $C(k) \neq \emptyset$. This shows the following proposition.

PROPOSITION 3.1. *Let $C$ be a smooth projective curve over a field $k$. Then $C$ is $k$-rational if and only if $p_g(C) = 0$ and $C(k) \neq \emptyset$.*

Equivalently, in dimension one, we have that $C$ is rational over $k$ if and only if $C$ is geometrically rational over $k$ and $C(k) \neq \emptyset$. In particular, this shows that rationality and stable rationality are equivalent in dimension one (and, over algebraically closed fields, are further equivalent to rational connectedness).

In general, the assumption that $C(k) \neq \emptyset$ is necessary: over any field $k$ with non-trivial 2-torsion in its Brauer group, there exist pointless conics over $k$. (For a concrete example, take $k = \mathbb{R}$ and $C$ defined by $x_0^2 + x_1^2 + x_2^2$.)

**3.2. Surfaces.** The rationality problem for surfaces is beautifully exposited in [**Has09**, Sections 2 and 3], [**VA13**, Sections 1 and 2], [**MT86**], and [**Kol96**, Section III.2]. Thus, in this section we will give an overview of some statements and refer the interested reader to these references for details and proofs.

DEFINITION 3.2. A smooth projective surface $S$ over a field $k$ is minimal if any birational morphism to a smooth surface $S \to S'$ over $k$ is an isomorphism. A $(-1)$-curve on a smooth projective surface $Y$ over $k$ is a smooth geometrically connected curve $E \subset Y$ of genus zero that is defined over $k$ and satisfies $E^2 = -1$.

THEOREM 3.3 (Castelnuovo's contraction theorem). *Let $S$ be a smooth projective surface over an algebraically closed field. Then $S$ is minimal if and only if $S$ contains no $(-1)$-curves. Furthermore, if $S$ contains a $(-1)$-curve $E$, then there exists a smooth projective surface $S'$ and a morphism $S \to S'$ contracting $E$ to a point, so that $S$ is the blow-up of $S'$ at this point.*

Thus, every smooth projective surface over an algebraically closed field is birationally equivalent to a minimal smooth projective surface, and furthermore can be obtained from one as a sequence of blow-ups. Therefore, to classify the rational surfaces over an algebraically closed field, it suffices to classify the minimal rational surfaces.

THEOREM 3.4. *Let $S$ be a smooth projective minimal surface over an algebraically closed field. Then $S$ is rational if and only if $S$ is isomorphic to either $\mathbb{P}^2$ or to a Hirzebruch surface $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 0$ or $e \geq 2$.*

Castelnuovo's rationality criterion for surfaces (proven in positive characteristic by Zariski [**Zar58a, Zar58b**]) characterizes rationality over an algebraically closed field using two invariants: the irregularity and second plurigenus:

THEOREM 3.5 (Castelnuovo's criterion for rationality). *Let $S$ be a smooth projective surface over an algebraically closed field. Then $S$ is rational if and only if $h^1(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(2K_S)) = 0$.*

For smooth surfaces over algebraically closed fields of characteristic zero, Castelnuovo's rationality criterion shows that rationality, stable rationality, unirationality, and rational connectedness are equivalent. This equivalence does not hold over algebraically closed fields of positive characteristic. Rationality and stable rationality are still equivalent by Castelnuovo's criterion, Proposition 2.5, and birational invariance of the cohomology of the structure sheaf (see [**CR11**]); however, Zariski constructed examples of non-rational surfaces admitting purely inseparable dominant rational maps from $\mathbb{P}^2$.

Now we turn to non-closed fields. If $S$ is a smooth projective surface over a field $k$ and $E$ is a $(-1)$-curve on $S$ defined over $k$, then contracting $E$ yields a birational morphism over $k$. However, these are not the only $(-1)$-curves that can be contracted.

EXAMPLE 3.6. Consider the geometric points $p = [i : 0 : 1]$ and $q = [-i : 0 : 1]$ of $\mathbb{P}^2_{\mathbb{Q}}$. Neither point is defined over $\mathbb{Q}$, but the Galois orbit $\{p, q\}$ is. Blowing up these two points yields a birational morphism $S \to \mathbb{P}^2_{\mathbb{Q}}$ over $\mathbb{Q}$.

The surface $S$ in the previous example does not contain any $(-1)$-curves defined over $\mathbb{Q}$, but it is not minimal. Over arbitrary fields, we have the following characterization of minimal surfaces.

THEOREM 3.7 (See [**Has09**, Theorem 3.2]). *Let $S$ be a smooth projective surface over a field $k$. Then $S$ is minimal if and only if $S_{\overline{k}}$ admits no Galois-invariant collection of pairwise disjoint $(-1)$-curves.*

If $S$ is not minimal, then contracting a Galois-invariant collection of pairwise disjoint $(-1)$-curves yields a birational morphism (defined over $k$) to a smooth projective surface $S'$ over $k$. Thus, the rationality question is reduced to considering minimal surfaces. If a minimal surface over a field $k$ is rational, then is it necessarily geometrically rational, and the geometrically rational minimal surfaces are classified as follows:

THEOREM 3.8 (Enriques, Manin, Iskovskikh [**Isk79**, Theorem 1]). *Let $S$ be a smooth projective minimal surface over a field $k$. Then $S$ is geometrically rational over $k$ if and only if $S$ is isomorphic to one of the following:*

(1) $\mathbb{P}^2$,
(2) *a smooth quadric $S \subset \mathbb{P}^3$ with* $\operatorname{Pic}(S) = \mathbb{Z}$,
(3) *a del Pezzo surface (meaning $-K_S$ is ample) with* $\operatorname{Pic}(S) = \mathbb{Z} \cdot K_S$, *or*
(4) *a conic bundle $S \to C$ over a smooth genus zero curve with* $\operatorname{Pic}(S) = \mathbb{Z} \oplus \mathbb{Z}$.

Here conic bundle means that the generic fiber of $f \colon S \to C$ is a smooth projective genus zero curve. Over an algebraically closed field, Tsen's theorem implies that $f$ has a section. In contrast, over non-closed fields, conic bundles may not in general have sections, as the following example shows.

EXAMPLE 3.9. Let $k \subset \mathbb{R}$ be any subfield of the real numbers. Let $S \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ be the smooth bidegree $(1, 2)$ (non-minimal) surface defined by $t_0 x_0^2 + t_1 x_1^2 + (t_0 - t_1) x_2^2$, where the $x_i$ are coordinates on $\mathbb{P}^2_k$ and the $t_i$ are coordinates on $\mathbb{P}^1_k$. Then the projection $f \colon S \to \mathbb{P}^1_{\mathbb{R}}$ is a conic bundle, which has three geometric singular fibers, which occur over the $k$-points $[0:1]$, $[1:0]$, and $[1:1]$ of $\mathbb{P}^1_k$. Over the point $[0:1]$, the fiber is defined by $x_1^2 - x_2^2$ and has two irreducible components over $k$. Over the points $[1:0]$ and $[1:1]$, the fiber is defined by $x_0^2 + x_1^2$ and is irreducible over $k$.

Over any point $[t_0 : t_1] \in \mathbb{P}^1(\mathbb{R})$ with $t_0 \neq 0$ and $0 < \frac{t_1}{t_0} < 1$, the fiber of $f$ is defined by a positive definite quadratic form and hence has no $k$-points. Therefore $f$ does not have a section defined over $k$.

From Theorem 3.8, the next natural question is to ask which of the geometrically rational surfaces are rational over the given ground field. In general, not every geometrically rational surface over $k$ is necessarily $k$-rational.

Example 3.10. Let $k$ be a field with non-trivial 2-torsion in its Brauer group, let $C$ be a pointless conic over $k$, and let $S = C \times C$. Then $S(k) = \emptyset$, so $S$ is not $k$-rational by Proposition 2.8. However, $S_{\overline{k}} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ is $\overline{k}$-rational.

In dimension one, we saw in Proposition 3.1 that the existence of a rational point is sufficient to guarantee rationality of a geometrically rational curve. However, starting from dimension two, this is no longer true. One source of such examples comes from B. Segre's theorem on rationality of cubic surfaces. If $S$ is a smooth cubic surface over a field $k$, then $S_{\overline{k}}$ is isomorphic to the blow-up of $\mathbb{P}^2_{\overline{k}}$ at six points, so $\mathrm{Pic}(S_{\overline{k}}) \cong \mathbb{Z}^7$. However, over $k$, the Picard rank $\rho(S)$ of $S$ could be strictly less than 7. If $S$ has Picard rank one over $k$, then $S$ is minimal.

Theorem 3.11 ([**Seg42**]). *Let $S$ be a smooth projective cubic surface over a field $k$. If $\rho(S) = 1$, then $S$ is not $k$-rational.*

The rough idea of the argument is to use the Noether–Fano method. This involves arguing that if such a cubic surface $S$ were birationally equivalent to $\mathbb{P}^2$ then $S$ must admit a highly singular linear system, and obtaining a contradiction from this. We refer the reader to [**KSC04**, Chapter 2] for the proof. B. Segre also constructed examples of surfaces satisfying the assumptions of Theorem 3.11:

Example 3.12 ([**Seg51a**]). Let $S \subset \mathbb{P}^3_{\mathbb{Q}}$ be the cubic surface defined by the equation $a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3$ for some $a_i \in \mathbb{Q}$ with the property that, for all permutations $\sigma$ of four letters,

$$\frac{a_{\sigma(0)}a_{\sigma(1)}}{a_{\sigma(2)}a_{\sigma(3)}} \notin \mathbb{Q}^3.$$

Then $\rho(S) = 1$, so $S$ is not $\mathbb{Q}$-rational by Theorem 3.11. On the other hand, many of these cubic surfaces contain $\mathbb{Q}$-points.

Here is an example over the real numbers of a smooth projective geometrically rational surface that has real points but is not rational over $\mathbb{R}$.

Example 3.13. Over the real numbers, let $S \subset \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}}$ be the smooth bidegree $(2,2)$ surface defined by

$$(3t_0 + t_1)(2t_0 + t_1)x_0^2 + (t_0 + t_1)t_1x_1^2 + (-t_0 + t_1)(-2t_0 + t_1)x_2^2.$$

The projection $f\colon S \to \mathbb{P}^1_{\mathbb{R}}$ is a conic bundle with six singular fibers over the points $[1:-3]$, $[1:-2]$, $[1:-1]$, $[1:0]$, $[1:1]$, and $[1:2]$; each of these singular fibers is defined by a quadratic form of signature $(2,0)$ and thus is irreducible over $\mathbb{R}$. A smooth fiber of $f$ over $[t_0 : t_1] \in \mathbb{P}^1(\mathbb{R})$ contains $\mathbb{R}$-points if and only if $t_0 \neq 0$

and $\frac{t_1}{t_0} \in (-3,-2) \cup (-1,0) \cup (1,2)$ (see Figure 1). This shows that $S(\mathbb{R})$ has three connected components (in the Euclidean topology), so $S$ is not $\mathbb{R}$-rational by Theorem 2.9. On the other hand, the conic bundle $f$ has a section over $\mathbb{C}$ by Tsen's theorem, so $S$ is geometrically rational by Example 2.2.

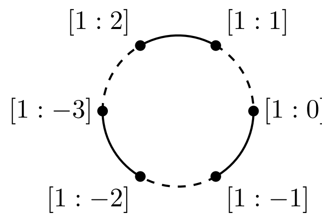



FIGURE 1. The fibers of the conic bundle $f\colon S \to \mathbb{P}^1_{\mathbb{R}}$ in Example 3.13 have no real points over the dashed regions in $\mathbb{P}^1(\mathbb{R})$.

These examples show that the rationality problem is more subtle in dimension two. Nonetheless, this question has a complete answer:

THEOREM 3.14 (B. Segre, Manin, Iskovskikh). *Let $S$ be a smooth projective minimal surface over a field $k$. Then $S$ is $k$-rational if and only if $S$ is geometrically rational over $k$, $K_S^2 \geq 5$, and $S(k) \neq \emptyset$.*

For a proof of Theorem 3.14, see [**Isk96**, Theorem 2.6 and page 642] over perfect fields and [**BW23**, Proposition 4.16] over imperfect fields.

The non-minimal surface $S$ in Example 3.9 has $k$-points and satisfies $K_S^2 = 5$, so Theorem 3.14 implies that $S$ is $k$-rational. This can be seen more explicitly as follows:

EXAMPLE 3.15. Let $S$ be the surface from Example 3.9, and let $S \to S'$ be the morphism obtained by contracting a component of the singular fiber over $[0 : 1]$. Then $S'$ is a conic bundle over $\mathbb{P}^1_k$ and satisfies $K_{S'}^2 = 6$. The surface $S'$ is a non-minimal del Pezzo surface by [**Isk79**, Theorems 4 and 5], and over $\overline{k}$ it contains a pair of Galois conjugate disjoint $(-1)$-curves [**Isk96**, page 614]. Contracting these gives a birational morphism $S' \to S''$ to a degree 8 del Pezzo surface $S''$ over $k$ with $\rho(S'') = 1$, which embeds in $\mathbb{P}^3$ as a smooth quadric (see [**VA13**, Section 2.2]). Since $S$ has a $k$-point, $S''$ does as well, so $S''$ is $k$-rational by Example 2.1. Therefore $S$ is $k$-rational.

As mentioned above, rationality is equivalent to stable rationality for surfaces over algebraically closed fields. However, over non-closed fields, Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer showed that these two notions are in general different.

EXAMPLE 3.16 ([**BCTSSD85**, Théorèmes 1 et 2]). Let $k$ be a field of characteristic $\neq 2$ that admits a Galois extension whose Galois group is the symmetric group $\mathfrak{S}_3$. Let $S$ be a smooth projective model of the affine surface defined in $\mathbb{A}^3_k$ by $y^2 - dz^2 = P(x)$ where $P(x)$ is an irreducible separable polynomial of degree 3 and $d \in k$ is its discriminant. If $d \notin k^2$, then $S$ is $k$-birational to a minimal surface $S'$ with $K_{S'}^2 \leq 4$, so $S$ is not $k$-rational by Theorem 3.14. However, [**BCTSSD85**] shows that $S \times \mathbb{P}^3_k$ is $k$-rational (this was later improved to $S \times \mathbb{P}^2_k$ in [**SB04**]), so $S$ is stably $k$-rational.

Over the real numbers, there is another characterization of rational surfaces using the topology of the locus of real points. B. Segre observed that a smooth cubic surface over the real numbers is necessarily $\mathbb{R}$-unirational, but it is not $\mathbb{R}$-rational if its real locus is disconnected [**Seg51b**]. Indeed, as we saw in Theorem 2.9, disconnectedness of the real locus is an obstruction to (stable) $\mathbb{R}$-rationality. Comessatti used this observation and the classification of complex surfaces to characterize $\mathbb{R}$-rationality in dimension two:

THEOREM 3.17 (Comessatti, see [**Sil89**, Chapter III.4]). *Let $S$ be a smooth projective surface over $\mathbb{R}$. The following are equivalent:*

(1) *$S$ is $\mathbb{R}$-rational;*
(2) *$S$ is stably $\mathbb{R}$-rational;*
(3) *$S_{\mathbb{C}}$ is $\mathbb{C}$-rational, and $S(\mathbb{R})$ is nonempty and connected.*

## 4. Threefolds

In this section, we consider threefolds. Over the complex numbers, the notions of rationality and unirationality are equivalent for smooth projective varieties of dimension at most two; however, in dimension three, these two notions differ in general. Unlike in lower dimensions, the rationality problem for threefolds over $\mathbb{C}$ does not yet have a complete answer, although many results are known. Over non-closed fields, there are additional subtleties in dimension three as well. For example, let $X$ be a smooth projective geometrically rational variety over a field $k$, and assume $X(k) \neq \emptyset$. If $\dim X \leq 2$, then the $k$-rationality of $X$ is completely understood using the Galois action on the geometric Picard group $\operatorname{Pic}(X_{\overline{k}})$; furthermore, if $k$ is imperfect and $X_{k'}$ is $k'$-rational for some purely inseparable extension $k'/k$, then $X$ is $k$-rational. Both statements are false in dimension three (see Section 4.2.1).

In Section 4.1, we will briefly overview three examples of unirational, non-rational complex threefolds and will indicate the rationality obstructions used for each of these. Then, in Section 4.2, we will survey what is currently known about rationality in dimension three, over both algebraically closed and non-closed fields, focusing on three classes of threefolds: Fano threefolds (Section 4.2.1), del Pezzo fibrations (Section 4.2.2), and conic bundles (Section 4.2.3). Finally, in Section 4.3, we will describe some rationality obstructions over non-closed fields that generalize the three obstructions over $\mathbb{C}$ mentioned in Section 4.1.

**4.1. Rationality, stable rationality, and unirationality of complex threefolds.** For smooth projective varieties over $\mathbb{C}$, the notions of rationality, stable rationality, unirationality, and rational connectedness are equivalent in dimensions one and two. It was a longstanding question, known as the Lüroth problem, to determine whether rationality and unirationality are equivalent in dimension three. This was answered in the negative in the 1970s by three examples using different methods. There are already many excellent surveys describing these methods in more detail (e.g., [**Bea16, Deb24**]), so here we will just give a brief summary of some aspects of each method.

(1) (Birational rigidity [**IM71**]) Let $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ be a smooth quartic threefold. Iskovskikh–Manin proved that the birational automorphism group $\operatorname{Bir}(X_4)$ is finite and therefore that $X_4$ is not rational (see Example 2.7).

On the other hand, some quartic threefolds were already known to be unirational, so their result gave examples of unirational, non-rational threefolds.

Stable rationality of these threefolds has also been studied. Using specialization methods, Colliot-Thélène–Pirutka proved that a very general quartic threefold is not stably rational [**CTP16**] (see [**Pir18**] for a survey on these techniques).

(2) (Brauer group [**AM72**]) Artin–Mumford proved that the torsion subgroup of $H^3(-,\mathbb{Z})$ is a stable birational invariant for smooth projective complex varieties. The example they constructed is a smooth threefold $X$ obtained as the resolution of a certain quartic double solid with ten nodes, and they proved that $H^3(X,\mathbb{Z})$ has nontrivial 2-torsion, so $X$ is not (stably) rational. On the other hand, they used the Grassmannian of lines in $\mathbb{P}^3_{\mathbb{C}}$ to give a unirationality construction for $X$, thus showing that $X$ is unirational.

The product $X \times \mathbb{P}^{n-3}_{\mathbb{C}}$ gives examples of varieties in any dimension $n \geq 3$ that are unirational but not (stably) rational over $\mathbb{C}$.

(3) (Intermediate Jacobian [**CG72**]) Let $X_3 \subset \mathbb{P}^4_{\mathbb{C}}$ be a smooth cubic threefold. Then $X_3$ is unirational, and Clemens–Griffiths proved that $X_3$ is not rational by proving that its intermediate Jacobian obstructs rationality. Briefly, the intermediate Jacobian is an abelian variety that "parametrizes 1-cycles" on the threefold $X_3$, and if $X_3$ were rational then its intermediate Jacobian would necessarily have a certain structure. This obstruction, which obstructs rationality but not stable rationality, will be discussed more in Section 4.3.3 below.

Recent work of Engel–de Gaay Fortman–Schreieder proves that a very general cubic threefold $X_3 \subset \mathbb{P}^4_{\mathbb{C}}$ is not stably rational [**EdS25**].

The first two methods can be modified to work in any dimension, whereas the method of intermediate Jacobians is specific to dimension three.

The example of Artin–Mumford shows that stable rationality does not in general coincide with unirationality in dimensions three and higher. It is also known that rationality and stable rationality differ in general, due to Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer. Over certain non-closed fields $k$, [**BCTSSD85**] constructed examples of stably rational, irrational surfaces over $k$ (Example 3.16). If $k$ is the function field in one variable over an algebraically closed field $k_0$ of characteristic $\neq 2$, then they furthermore used these surfaces over $k$ to construct examples over $k_0$ of threefolds that are stably rational but not rational over $k_0$. (The rationality obstruction they use is the intermediate Jacobian of the threefold.)

**4.2. Rationality of threefolds over fields.** First we work over algebraically closed fields. Let $V'$ be a smooth projective threefold over an algebraically closed field. Assume $K_{V'}$ is not pseudoeffective (e.g., this holds if $V'$ is rational) and the characteristic is not 2 or 3. Then by the Minimal Model Program (see, e.g., [**Mor88, BW17, HW22**]), $V'$ is birationally equivalent to a projective variety $V$ (with $\mathbb{Q}$-factorial terminal singularities) that admits the structure of a Mori fiber space $\pi\colon V \to Z$. This means that $\pi$ is a projective contraction to a normal projective variety $Z$ with $\dim Z < \dim V$, $-K_V$ is $\pi$-ample, and $\rho(V/Z) = 1$. We have three possibilities for Mori fiber spaces in dimension three:

- If $\dim Z = 0$, then $V$ is a Fano threefold.
- If $\dim Z = 1$, then $V$ is a del Pezzo fibration over a curve.
- If $\dim Z = 2$, then $V$ is a conic bundle over a surface.

Now let $k$ be any field. Motivated by the above, we will consider the rationality problem for the following three classes of threefolds:

(1) Smooth Fano threefolds (i.e., $X$ with $-K_X$ ample) with $\rho(X) = 1$;
(2) del Pezzo fibrations $f\colon X \to \mathbb{P}^1_k$ with $X$ smooth and $\rho(X/\mathbb{P}^1_k) = 1$;
(3) Conic bundles $\pi\colon X \to \mathbb{P}^2$ with $X$ smooth and $\rho(X/\mathbb{P}^2_k) = 1$.

To motivate (2), if $f\colon X \to C$ is a del Pezzo fibration over a smooth projective curve and $X$ is $k$-rational, then $C$ is $k$-unirational so $C \cong \mathbb{P}^1_k$. To motivate (3), if $k$ has characteristic zero, $\pi\colon X \to S$ is a conic bundle over a smooth projective surface, and $X$ is $k$-rational, then $S$ is geometrically rational over $k$ and $S(k) \neq \emptyset$. The simplest such surface is $\mathbb{P}^2_k$.

Question 4.1. Let $k$ be a field, and let $X$ be a smooth projective threefold over $k$ satisfying (1), (2), or (3). Is $X$ $k$-rational?

As in Section 3.2, we will first discuss geometric rationality over $\overline{k}$ and then turn to working over $k$.

4.2.1. *Fano threefolds.* First we consider one special family of Fano threefolds. If $k$ is any field and $X \subset \mathbb{P}^5_k$ is a smooth complete intersection of two quadrics, then $X$ is geometrically rational over $k$ by Example 2.3 and satisfies $\rho(X) = \rho(X_{\overline{k}}) = 1$. Many rationality obstructions vanish for this family; for example, the Brauer group (Section 4.3.2) of $X$ is trivial, and the absolute Galois group of $k$ acts trivially on $\mathrm{Pic}(X_{\overline{k}})$. In Example 2.3, we saw that if $X$ contains a line over $k$, then projection from this line shows that $X$ is $k$-rational. The following result of Hassett–Tschinkel and Benoist–Wittenberg shows that this is the *only* rationality construction for $X$:

Theorem 4.2 ([**HT21b**, Theorem 36], [**BW23**, Theorem A]). *Let $k$ be a field, and let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics over $k$. The following are equivalent:*

(1) *$X$ is $k$-rational;*
(2) *$X$ contains a line defined over $k$;*
(3) *The intermediate Jacobian torsor obstruction vanishes for $X$ over $k$.*

The intermediate Jacobian torsor (IJT) obstruction is a refinement of the Clemens–Griffiths obstruction over non-closed fields, using certain torsors under the intermediate Jacobian. This obstruction will be discussed more in Section 4.3.3. (We emphasize that this obstruction and result are specific to dimension three. For example, in any even dimension, there are examples of smooth complete intersections of two quadrics that are $\mathbb{R}$-rational but do not contain real lines [**HKT22**, Propositions 5.1 and 6.1].)

As one application of Theorem 4.2, Benoist–Wittenberg constructed an example of a threefold over an imperfect field whose rationality properties change under a purely inseparable extension. Such examples cannot exist in lower dimensions, see [**BW23**, Proposition 4.16].

Example 4.3 ([**BW23**, Theorem 4.14]). Let $k_0$ be an algebraically closed field of characteristic 2, and let $a, b, c \in k_0$ be pairwise distinct elements. Let $k = k_0((t))$ be the field of Laurent series over $k_0$, and let $X \subset \mathbb{P}^5_k$ be the smooth complete intersection of two quadrics defined by

$$tx_0x_1+x_2x_3+x_4x_5 = t(x_0^2+ax_0x_1+x_1^2)+(x_2^2+bx_2x_3+x_3^2)+(x_4^2+cx_4x_5+x_5^2) = 0.$$

Then $X$ has a $k$-point, and $X$ is not $k$-rational, but its base change becomes rational over the purely inseparable extension $k_0((t^{1/2}))$ of $k$.

Now let $X$ be a smooth projective Fano threefold over a field $k$ of characteristic zero. Much is understood about the geometric rationality of $X$, due to the classification of the 105 deformation families of smooth Fano threefolds over $\mathbb{C}$ by Iskovskikh and Mori–Mukai (see [**Isk77, Isk78, MM82**] and [**IP99**, Chapter 12]). Using the classification, Kuznetsov–Prokhorov studied the $k$-rationality problem for Fano threefolds with $\rho(X) = 1$. First, if the geometric Picard rank is one, they gave a full characterization of $k$-rationality.

THEOREM 4.4 ([**KP23**, Theorem 1.1]). *Let $X$ be a smooth projective Fano threefold over a field $k$ of characteristic zero. Assume $X_{\overline{k}}$ is $\overline{k}$-rational and $\rho(X_{\overline{k}}) = 1$.*

(1) *If $X_{\overline{k}}$ is not in Family №1.8, №1.9, or №1.14, then $X$ is $k$-rational if and only if $X(k) \neq \emptyset$.*
(2) *If $X_{\overline{k}}$ is in Family №1.8, №1.9, or №1.14, then $X$ is $k$-rational if and only if $X(k) \neq \emptyset$ and the IJT obstruction vanishes for $X$ over $k$.*

In the above, the numbers refer to the numbering used in the Mori–Mukai classification of deformation families. Family №1.14 is that of complete intersections of two quadrics and was addressed by Hassett–Tschinkel and Benoist–Wittenberg (Theorem 4.2).

Theorem 4.4 addresses the case when the geometric Picard rank of $X$ is one. However, it could happen that $X$ has Picard rank one over the field $k$, but its geometric Picard rank over $\overline{k}$ is higher:

EXAMPLE 4.5 ([**FM25**, Proposition 3.15]). The intersection of the three bidegree $(1,1)$ divisors in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$ given by

$$x_0y_1 + x_1y_0 - \sqrt{2}x_2y_2, \quad x_0y_2 + x_2y_0 - \sqrt{2}x_3y_3, \quad x_0y_3 + x_3y_0 - \sqrt{2}x_1y_1.$$

defines a smooth complex Fano threefold in Family №2.12 from the Mori–Mukai classification. This threefold has a real form $X$ with Picard rank one (over $\mathbb{R}$) and $\rho(X_{\mathbb{C}}) = 2$.

EXAMPLE 4.6 ([**KP23**, Theorem 1.4]). Let $H \subset \mathfrak{S}_4$ be a subgroup containing the Klein four-group. Let $k_0$ be a field of characteristic zero such that there is a surjection $G_{k_0} \to H$, and let $k = k_0(t)$ be the function field in one variable over $k_0$. Then there exists a smooth projective Fano threefold $X$ over $k$ such that

- $\rho(X) = 1$ and $X(k) \neq \emptyset$,
- $X_{\overline{k}}$ is $\overline{k}$-rational and $\rho(X_{\overline{k}}) = 4$, and
- $X$ is not (stably) $k$-rational.

In this case $X_{\overline{k}}$ belongs to Family №4.1 in the Mori–Mukai classification.

Examples 4.5 and 4.6 show that there exists fields $k$ of characteristic zero and $k$-forms $X$ of Fano threefolds in Families №2.12 and №4.1 that are geometrically rational over $k$ and satisfy $\rho(X_{\overline{k}}) > \rho(X) = 1$. Kuznetsov–Prokhorov studied the rationality problem for such Fano threefolds, and gave an almost complete characterization of $k$-rationality when $\rho(X) = 1$ and $\rho(X_{\overline{k}}) > 1$:

THEOREM 4.7 ([**KP23**, Theorems 1.1 and 1.2]). *Let $X$ be a smooth projective Fano threefold over a field $k$ of characteristic zero. Assume $X_{\overline{k}}$ is $\overline{k}$-rational, $\rho(X) = 1$, and $\rho(X_{\overline{k}}) > 1$. Then there are six possible deformation families for $X_{\overline{k}}$, and $k$-rationality is completely understood except in Family №4.1:*

(1) *If $X_{\overline{k}}$ is in Family №2.12, then $X$ is never $k$-rational.*
(2) *If $X_{\overline{k}}$ is in one of the four remaining deformation families, then $X$ is $k$-rational if and only if $X(k) \neq \emptyset$.*

Thus, in characteristic zero, the rationality problem for Fano threefolds with $\rho(X) = 1$ is understood *except* when $X_{\overline{k}}$ is in Family №4.1. In this last remaining case, Kuznetsov–Prokhorov conjecture that $X$ is never $k$-rational [**KP24**, Conjecture 1.3].

REMARK 4.8. In positive characteristic, Tanaka recently completed the classification of smooth Fano threefolds over algebraically closed fields [**Tan23**]. It seems reasonable to expect that many of the above results in characteristic zero can be extended to positive characteristic.

As for the question of stable rationality, a very general smooth Fano threefold over $\mathbb{C}$ is either rational or is not stably rational, see [**HT19, EdS25**]. Over non-closed fields, much less is currently known, even for geometrically rational Fano threefolds. For example, over the real numbers, there are smooth complete intersections of two quadrics $X \subset \mathbb{P}^5_{\mathbb{R}}$ that are not stably $\mathbb{R}$-rational because $X(\mathbb{R})$ is disconnected (see [**HT21b**, Section 11.4]). However, there are also examples that have connected real locus and yet are not $\mathbb{R}$-rational; the stable $\mathbb{R}$-rationality of these is currently an open question. Over real-closed fields, recent work of Colliot-Thélène–Pirutka–Scavia gives a negative answer to the analogous question.

EXAMPLE 4.9 ([**CPS25**, Théorème 3.6]). Let $R = \bigcup_{n \geq 1} \mathbb{R}((t^{1/n}))$ be the field of real Puiseux series. Let $q_0, q_1 \in \mathbb{R}[x_0, x_1, x_2, x_3, x_4, x_5]$ be quadratic forms defining a smooth complete intersection of two quadrics in $\mathbb{P}^5_{\mathbb{R}}$, and let $X \subset \mathbb{P}^5_R$ be the smooth complete intersection of two quadrics defined by

$$x_0^2 + x_1^2 + x_2^2 - x_2x_3 + tq_0 = x_4^2 + x_5^2 - x_1x_3 + tq_1 = 0.$$

Then $X(R)$ is nonempty and semi-algebraically connected, and $X$ is not stably $R$-rational.

4.2.2. *del Pezzo fibrations.* Let $f\colon X \to \mathbb{P}^1_k$ be a del Pezzo fibration over a field $k$, and assume the generic fiber of $f$ is a smooth del Pezzo surface of degree $d$.

First we discuss the situation over the algebraic closure. If $d \geq 5$, then $X$ is geometrically rational over $k$. Indeed, for any $d$, the function field $\overline{k}(t)$ is a $C_1$ field by Tsen's theorem, so $f$ admits a section over $\overline{k}$ [**CT87**, Proposition 2]. If $d \geq 5$, then the generic fiber of $f_{\overline{k}}$ is $\overline{k}(t)$-rational by Theorem 3.14, so $X_{\overline{k}}$ is $\overline{k}$-rational.

For $d \leq 4$, the rationality problem is more subtle and is not fully answered, even over the complex numbers. There are several results over $\mathbb{C}$ involving the construction of standard models (e.g., [**Cor96, AFK22**]) and birational rigidity (e.g., [**Puk98**]). For example, when $d = 4$, a result of Alexeev and Shramov gives necessary and sufficient condition for rationality in terms of the topological Euler characteristic of $X$ [**Shr06**]. Additionally, stable non-rationality is known for many del Pezzo fibrations of degree $d \leq 4$, see [**HT19, KO20**]. We refer the reader to

[**Kaw24**, Chapter 8] for additional references and more on what is known about rationality in this setting.

Over non-closed fields, one special class of del Pezzo fibrations that has been the subject of much recent study is that of quadric surface fibrations over $\mathbb{P}^1$ (in characteristic $\neq 2$). As mentioned above, such threefolds are always geometrically rational because the fibration admits a section over $\overline{k}$, but in general such a section may or may not descend to $k$. The (stable) $k$-rationality of such threefolds has been studied in [**FJS⁺24b, JJ24, FJS⁺24a, JS25, CP24, BP24, CPS25**], especially over $\mathbb{R}$ and more generally real closed fields, yet much remains unknown. For instance, the following problem is still open in general:

QUESTION 4.10. Over the real numbers, let $f\colon X \to \mathbb{P}^1_{\mathbb{R}}$ be a quadric surface fibration such that $X(\mathbb{R})$ is nonempty and connected, and assume $f$ does not have a section (over $\mathbb{R}$). Is $X$ $\mathbb{R}$-rational?

In some cases, the answer to Question 4.10 is known. For instance, if $f$ has $\geq 6$ singular geometric fibers, each of which is geometrically integral, and if the fiber of $f$ over some $p \in \mathbb{P}^1(\mathbb{R})$ contains real lines, then Wittenberg has recently shown that $X$ is not $\mathbb{R}$-rational, using the method of intermediate Jacobian torsors.

4.2.3. *Conic bundles.* Let $k$ be a field of characteristic $\neq 2$, and let $\pi\colon X \to \mathbb{P}^2_k$ be a conic bundle with $X$ smooth. Before stating some results, we will first need some definitions (see, e.g., [**Sar82, Pro18**]).

DEFINITION 4.11. We say $\pi\colon X \to \mathbb{P}^2_k$ is standard if $\rho(X/\mathbb{P}^2_k) = 1$.

The generic fiber of $\pi$ is smooth, and the discriminant of the conic bundle $\pi\colon X \to S$ is the divisor $\Delta \subset S$ parametrizing singular fibers of $\pi$. The divisor $\Delta$ is reduced and has at worst normal crossings singularities. The Stein factorization of the relative variety $\mathcal{F}_1(X_\Delta/\Delta)$ of lines in the fibers of $\pi$ over $\Delta$ defines a double cover $\varpi\colon \tilde{\Delta} \to \Delta$, and if $\Delta$ is smooth then $\varpi$ is étale.

Over algebraically closed fields, the rationality of standard conic bundles is well studied. We refer the reader to [**Pro18**] for a survey on these results. In particular, the rationality behavior is determined by the discriminant (and its double cover):

THEOREM 4.12 (Iskovskikh, Beauville, Tyurin, Masiewicki, Panin. See [**Pro18**, Corollary 3.9.1 and Theorem 9.1]). *Over an algebraically closed field of characteristic $\neq 2$, let $X \to \mathbb{P}^2$ be a standard conic bundle with smooth discriminant curve $\Delta$. Then $\deg \Delta \geq 3$, and the following hold.*

(1) *If $3 \leq \deg \Delta \leq 4$, then $X$ is rational.*
(2) *If the field is $\mathbb{C}$ and $\deg \Delta = 5$, then $X$ is rational if and only if $\varpi\colon \tilde{\Delta} \to \Delta$ is defined by an even theta characteristic.*
(3) *If $\deg \Delta \geq 6$, then $X$ is not rational.*

In part (2) above, the condition on the theta characteristic is important because not all conic bundles with $\deg \Delta = 5$ are rational. Namely, the blow-up of a smooth cubic threefold along a line exhibits a standard conic bundle structure $X \coloneqq \mathrm{Bl}_\ell(X_3) \to \mathbb{P}^2_{\mathbb{C}}$ with degree 5 discriminant; in this case $X_3$ (and hence $X$) is not rational due to its intermediate Jacobian, by the work of Clemens–Griffiths [**CG72**], and $\varpi$ is defined by an odd theta characteristic. The non-rationality result when $\deg \Delta \geq 6$ is due to Beauville, who used the method of intermediate Jacobians [**Bea77**]. Furthermore, for standard conic bundles over Hirzebruch surfaces over

$\mathbb{C}$, Shokurov proved that the intermediate Jacobian obstruction characterizes rationality [**Sho83**]. This answers the rationality question for standard conic bundles over minimal rational surfaces over $\mathbb{C}$.

Stable rationality is also well studied over algebraically closed fields. For standard conic bundles $X \to \mathbb{P}^2$ with discriminant $\Delta$, it suffices by Theorem 4.14 to consider the cases when $\deg \Delta \geq 5$. The following stable non-rationality result follows from a result of Hassett–Kresch–Tschinkel.

THEOREM 4.13 (Special case of [**HKT16**, Theorem 1]). *Over an algebraically closed field of characteristic $\neq 2$, let $X \to \mathbb{P}^2$ be a standard conic bundle with discriminant curve $\Delta$. If $\Delta$ is very general and* $\deg \Delta \geq 6$*, then $X$ is not stably rational.*

The remaining case of degree 5 discriminant has been studied over $\mathbb{C}$. If the double cover $\varpi\colon \tilde{\Delta} \to \Delta$ is defined by an even theta characteristic, then $X$ is rational by Theorem 4.14(2). If $\varpi$ is defined by an odd theta characteristic, then Tregub showed that $X$ is birational to a smooth cubic threefold [**Tre90**], and [**EdS25**] proved that a very general complex cubic threefold is not stably rational.

Now we turn to the situation over non-closed fields. If $\pi\colon X \to \mathbb{P}^2_k$ is a standard conic bundle over a non-closed field $k$, then $\rho(X/\mathbb{P}^2_k) = 1$, and the relative geometric Picard number $\rho(X_{\overline{k}}/\mathbb{P}^2_{\overline{k}})$ is at least one. If $\rho(X_{\overline{k}}/\mathbb{P}^2_{\overline{k}}) = 1$, then $\pi$ is geometrically standard over $k$.

For conic bundles that are standard but not geometrically standard over a field $k$, the rationality problem was studied in [**BW20, BP24**]. Over $\mathbb{R}$, [**BW20**] used intermediate Jacobians over non-closed fields to give examples over $\mathbb{R}$ of geometrically rational conic bundles that are not $\mathbb{R}$-rational but for which all previously known rationality obstructions vanish. [**BP24**] gave additional examples using birational rigidity techniques (see Example 4.18). Over the field of real Puiseux series, [**CPS25**] used specialization methods to give stably non-rational examples.

For geometrically standard conic bundles, by Theorem 4.12 it suffices to consider the case of $3 \leq \deg \Delta \leq 5$. When $\deg \Delta \leq 4$, partial results are known:

THEOREM 4.14 ([**FJS⁺24b**, Proposition 8.1 and Theorem 1.5]). *Let $k$ be a field of characteristic $\neq 2$, and let $X \to \mathbb{P}^2_k$ be a geometrically standard conic bundle with smooth discriminant curve $\Delta$. Then* $\deg \Delta \geq 3$*, and the following hold.*

(1) *If* $\deg \Delta = 3$ *and $X(k) \neq \emptyset$, then $X$ is $k$-rational.*
(2) *If* $\deg \Delta = 4$ *and $\tilde{\Delta}(k) \neq \emptyset$, then $X$ is $k$-rational.*
(3) *If* $\deg \Delta = 4$ *and* $(\operatorname{Br} k)[2] = 0$*, then $X$ is $k$-rational if and only if the IJT obstruction vanishes over $k$.*

Parts (1) and (2) of Theorem 4.14 use a modification of Iskovskikh's proof of Theorem 4.12(1), whereas part (3) uses the method of intermediate Jacobian torsors. The $\deg \Delta = 4$ case was further studied over the real numbers in [**JJ24, FJS⁺24a**], but a complete answer to the rationality question is still currently out of reach, even over $\mathbb{R}$. For instance, the (stable) $\mathbb{R}$-rationality of the following threefold conic bundle is currently unknown.

EXAMPLE 4.15 ([**JJ24**, Remark 6], cf. [**FJS⁺24b**, Example 1.6]). Let $X$ be the double cover of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}}$ defined by

$$w^2 = t_0^2(-x_0^2 - x_1^2 + x_2^2) + 2t_0t_1(x_0^2 - 3x_1^2 - x_2^2) + t_1^2(-10x_0^2 - 10x_1^2 - x_2^2).$$

The second projection $\pi_2 \colon X \to \mathbb{P}^2_{\mathbb{R}}$ endows $X$ with the structure of a geometrically standard conic bundle with smooth quartic discriminant $\Delta$, and the first projection $\pi_1 \colon X \to \mathbb{P}^2_{\mathbb{R}}$ is a quadric surface fibration satisfying the hypotheses of Question 4.10. The threefold $X$ has no IJT obstruction, no unramified cohomology obstruction, and no real topological obstruction to rationality.

**4.3. Some rationality obstructions over non-closed fields.** In this section, we give an overview of some results generalizing the three obstructions discussed in Section 4.1 from $\mathbb{C}$ to other fields.

4.3.1. *Birational rigidity.* This section will involve some mildly singular varieties, as required by the MMP. Let $X$ be a terminal, $\mathbb{Q}$-factorial Fano variety over a field $k$, and assume $\rho(X) = 1$ (so that $X \to \operatorname{Spec} k$ is a Mori fiber space).

Definition 4.16. A Fano variety $X$ as above is said to be birationally rigid over $k$ if for any birational equivalence $\phi \colon X \dashrightarrow X'$ to the total space $X'$ of a Mori fiber space over $k$ (with $\mathbb{Q}$-factorial terminal singularities), $X$ is $k$-isomorphic to $X'$.

If furthermore every such birational map $\phi$ is required to be a $k$-isomorphism, then $X$ is said to be birationally superrigid over $k$.

That is, $X$ is birationally superrigid if it is birationally rigid and $\operatorname{Aut}(X) = \operatorname{Bir}(X)$. Birational rigidity is an obstruction to rationality because for any $n \geq 2$, $\mathbb{P}^n_k$ is birationally equivalent but not isomorphic to the total space of the Mori fiber space $\mathbb{P}^{n-1}_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$.

Both notions have been primarily studied over $\mathbb{C}$. As mentioned in Section 4.1. Iskovskikh–Manin proved for smooth quartic threefolds $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ that $\operatorname{Bir}(X_4)$ is finite [**IM71**]. To do this, they proved that every birational automorphism of $X_4$ is a regular automorphism of $X_4$; in fact, their methods show that $X_4$ is birationally superrigid (see [**Che05**, Section 2.1]). This technique uses what is known as the Noether–Fano method or the method of maximal singularities. Very roughly, the idea is as follows. If $X \dashrightarrow X'$ is a nontrivial birational equivalence, then there is a movable linear system $|M| \subset |-mK_X|$ for some $m \geq 1$ such that the movable log pair $(X, \frac{1}{m}|M|)$ is highly singular. So, to show that $X$ is birationally superrigid, one shows that such singularities do not occur. For more details, see the excellent surveys [**Che05, Puk13**] and [**KSC04**, Chapter 5].

Birational rigidity techniques have also been applied over non-closed fields, primarily in dimension two, e.g., in Segre's proof of non-rationality of cubic surfaces with Picard rank one (in fact, such surfaces are birationally rigid, see, e.g., [**Che05**, Section 1.5]). The property of birational rigidity may not in general behave well under field extensions, since the property of being a Mori fiber space need not be preserved. For example, if $X$ is a smooth cubic surface over $k$ with $\rho(X) = 1$ (e.g., Example 3.12), then $X$ is birationally rigid over $k$ but $X_{\overline{k}}$ is $\overline{k}$-rational and $\rho(X_{\overline{k}}) = \mathbb{Z}^{\oplus 7}$. The following question of Kollár is currently open:

Question 4.17 ([**Kol09**, Question 4]). Let $X$ be a Fano variety over a field $k$. If $X_{\overline{k}}$ is birationally rigid over $\overline{k}$, is $X$ birationally rigid over $k$?

Birational (super)rigidity can also be defined for Mori fiber spaces whose total space is not necessarily Fano. In this setting, in dimension three, Benoist–Pirutka recently gave the first application of birational rigidity to prove non-rationality of threefolds over non-closed fields. By extending Sarkisov's result on superrigidity of standard conic bundles with high degree discriminant to non-closed fields ([**Sar82**, Main Theorem 4.1], [**BP24**, Theorem 0.5]), they proved non-rationality of certain standard (but not geometrically standard) conic bundles over non-closed fields:

Example 4.18 ([**BP24**, Theorem 0.4]). Let $R$ be a real closed field (e.g., $\mathbb{R}$), and let $\pi\colon X \to S$ be a smooth projective model of the conic bundle over $\mathbb{P}^2_R$ (with coordinates $v, w$) given by $x^2+y^2 = f(v,w)$ where $f \in R[v,w]$ has even degree $\geq 12$ and defines a nodal rational curve. Then the threefold $X$ is not $R$-rational, but it is geometrically rational (in fact, $\pi$ admits a rational section over $\overline{R}$). Furthermore, if the polynomial $f$ is nonnegative or if it defines a curve with smooth $R$-points, then the locus $X(R)$ is nonempty and semi-algebraically connected.

4.3.2. *Unramified cohomology.* In this section, we will briefly state some results about unramified cohomology, which was first introduced by Colliot-Thélène and Ojanguren [**CTO89**]. There are many excellent surveys on unramified cohomology (e.g., [**Sch21, CT95**], [**Pir18**, Section 3], [**CTS07**, Section 5]), so we will refer the interested reader to these references for more details.

Let $X$ be a smooth projective variety over a field $k$, let $m \geq 1$ be an integer that is invertible in $k$, and let $i, j \geq 1$ be integers. For each discrete valuation $v$ of the function field $k(X)$ that is trivial on $k$, there is a residue map of Galois cohomology groups

$$\delta^i_v\colon H^i\left(k(X), \mu_m^{\otimes j}\right) \to H^{i-1}\left(\kappa(v), \mu_m^{\otimes (j-1)}\right).$$

The following is not the original definition of unramified cohomology, but it is equivalent since $X$ is smooth and projective:

Definition 4.19. The $i$-th unramified cohomology group of $k(X)/k$ with coefficients in $\mu_m^{\otimes j}$ is the subgroup

$$H^i_{nr}(k(X)/k, \mu_m^{\otimes j}) \subset H^i(k(X), \mu_m^{\otimes j})$$

of elements in the kernel of $\delta_v$ for any discrete valuation $v$ of $k(X)$ arising from a prime divisor on the smooth projective variety $X$.

In the case of $i = 2$ and $j = 1$, there is an isomorphism

$$H^2_{nr}(k(X)/k, \mu_m) \cong (\operatorname{Br} X)[m]$$

where $\operatorname{Br} X \coloneqq H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the (cohomological) Brauer group of $X$. This is a torsion group, and $(\operatorname{Br} X)[m]$ denotes its $m$-torsion.

Unramified cohomology provides an obstruction to (stable) rationality:

Theorem 4.20 (Colliot-Thélène–Ojanguren). *If $X$ and $Y$ are smooth projective varieties over $k$ such that $X$ is birationally equivalent over $k$ to $Y \times \mathbb{P}^r_k$ for some $r \geq 0$, then for all $m, i, j \geq 1$ with $m$ invertible in $k$,*

$$H^i_{nr}(k(X)/k, \mu_m^{\otimes j}) \cong H^i_{nr}(k(Y)/k, \mu_m^{\otimes j}).$$

In particular, if $X$ is smooth, projective, and stably $k$-rational, then $H^i(k, \mu_m^{\otimes j}) \cong H^i_{nr}(k(X)/k, \mu_m^{\otimes j})$ for all $i \geq 1$. This powerful obstruction has been used to great effect, especially in combination with specialization methods, to prove (stable) non-rationality of many varieties (see, e.g., [**Sch25**, Section 3]).

Unramified cohomology generalizes the Artin–Mumford obstruction mentioned in Section 4.1. Recall that Artin–Mumford showed that the torsion subgroup of $H^3(V, \mathbb{Z})$ is a stable birational invariant for smooth projective complex varieties $V$.

If $H^2(V,\mathcal{O}_V)=0$ (e.g., if $V$ is rationally connected), then the torsion subgroup of $H^3(V,\mathbb{Z})$ is isomorphic to $\operatorname{Br} V$. See [**Bea16**, Section 6.2] for details.

Over the real numbers, unramified cohomology is related to the real topology of a variety. Let $s$ denote the number of connected components of the real locus of a smooth projective real variety. In dimension one, Witt proved that if $C$ is a smooth projective geometrically connected curve over $\mathbb{R}$, then $\operatorname{Br} C \cong (\mathbb{Z}/2)^{\oplus s}$. In dimension two, a result of Silhol together with the Hochschild–Serre spectral sequence implies that if $S$ is a smooth projective geometrically rational surface over $\mathbb{R}$, then $\operatorname{Br} S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus(2s-1)}$ (see [**BW20**, Proof of Theorem 5.7]). Since $\operatorname{Br}\mathbb{R}\cong\mathbb{Z}/2\mathbb{Z}$, this gives another perspective on the real topological obstruction we saw earlier (Theorem 2.9). In any dimension, Colliot-Thélène–Parimala proved:

Theorem 4.21 ([**CTP90**, Main theorem]). *Let $X$ be a smooth projective geometrically connected variety over $\mathbb{R}$, and let $s$ be the number of connected components of $X(\mathbb{R})$. Then $H^i_{nr}(\mathbb{R}(X)/\mathbb{R},\mathbb{Z}/2)\cong(\mathbb{Z}/2)^{\oplus s}$ for any $i>\dim X$.*

For curves, $\operatorname{Br} C\cong H^2_{nr}(\mathbb{R}(C)/\mathbb{R}),\mathbb{Z}/2)$, so this generalizes Witt's result to any dimension, see [**CTP90**, first paragraph]).

For certain classes of threefolds (e.g., geometrically standard conic bundles over $\mathbb{P}^2_{\mathbb{R}}$), connectedness of the real locus is sufficient to ensure that the unramified cohomology obstruction vanishes [**BW20**, Theorem 1.4 and following paragraphs]. However, this is not true for all threefolds, even among geometrically rational ones:

Example 4.22 ([**BW20**, Theorem 5.7]). Let $S\to\mathbb{P}^1_{\mathbb{R}}$ be a smooth, projective, relatively minimal conic bundle given by the equation $x_0^2+x_1^2-(t-t^3)x_2^2$ in $\mathbb{P}^2_{\mathbb{R}}\times\mathbb{A}^1_{\mathbb{R}}$, and let $X$ be the smooth projective threefold such that $X\to S$ is the standard (but not geometrically standard) conic bundle given by the quaternion algebra $(-1,t-t^2)\in\operatorname{Br}\mathbb{R}(S)$. Since $S(\mathbb{R})$ has two connected components, $\operatorname{Br} S\cong(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

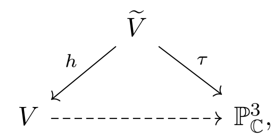
Benoist–Wittenberg use this to prove that $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}\subset(\operatorname{Br} X)[2]$, so $\operatorname{Br}\mathbb{R}\ncong\operatorname{Br} X$ and hence $X$ is not (stably) $\mathbb{R}$-rational. On the other hand, they also prove that $X_{\mathbb{C}}$ is $\mathbb{C}$-rational, $X(\mathbb{R})$ is nonempty and connected, and the intermediate Jacobian of $X$ is trivial.

4.3.3. *Intermediate Jacobian (torsor) obstructions.* In this section, we first review the Clemens–Griffiths intermediate Jacobian obstruction to rationality over $\mathbb{C}$ and discuss generalizations of intermediate Jacobians over other fields. Then, over non-closed fields, we describe the intermediate Jacobian torsor obstruction (Definition 4.25) of Hassett–Tschinkel and Benoist–Wittenberg and give examples of applications.

First we recall some results over $\mathbb{C}$. For more details and references, see, e.g., [**Bea16**, Section 3] and [**Voi16**, Section 3]. Over the complex numbers, the intermediate Jacobian of a smooth projective rationally connected threefold $V$ is a principally polarized abelian variety $J^3(V)$ defined using Hodge theory. The intermediate Jacobian is equipped with an Abel–Jacobi map $\mathrm{CH}^2(V)_{alg}\to J^3(V)$ from the subgroup $\mathrm{CH}^2(V)_{alg}\subset\mathrm{CH}^2(V)$ of algebraically trivial 1-cycle classes. By a result of Bloch–Srinivas, this Abel–Jacobi map is an isomorphism. The following result of Clemens–Griffiths is known as the intermediate Jacobian obstruction to rationality.

Theorem 4.23 ([**CG72**, Corollary 3.26]). *If $V$ is rational, then $J^3(V)$ is isomorphic, as principally polarized abelian varieties, to a product of Jacobians of smooth curves.*

To give a rough idea of why Theorem 4.23 is true, if $V$ is rational, then there is a birational map $V \dashrightarrow \mathbb{P}^3_{\mathbb{C}}$. This can be resolved as




where $h$ is birational and projective, and $\tau$ is a series of blow-ups along smooth curves and points. The curves whose Jacobians appear in Theorem 4.23 are some of the centers of these blow-ups. (See [**Voi16**, Section 3.3.1] for details.)

Now let $V$ be a smooth projective rationally connected threefold over a field $k$. If $k$ is algebraically closed, Murre gave a different, cycle-theoretic construction of an abelian variety $\mathrm{Ab}^2(V)$ [**Mur85**], which agrees with the Hodge theoretic definition of $J^3(V)$ if $k = \mathbb{C}$. This abelian variety is equipped with a morphism $\mathrm{CH}^2(V)_{alg} \to \mathrm{Ab}^2(V)$ that satisfies a certain universal property, and $\mathrm{Ab}^2(V)$ provides an obstruction to rationality [**Bea77**, Proposition 4.6]. If $k$ is a perfect field, then [**ACMV17**] proved that $\mathrm{Ab}^2(V_{\overline{k}})$ descends to $k$.

Now assume $X$ is a smooth projective geometrically rational threefold over any field $k$. In this setting, Benoist–Wittenberg [**BW23**] constructed a certain group scheme $\mathbf{CH}^2_{X/k}$ over $k$, which we will refer to as the codimension 2 Chow scheme. We will now indicate a few ideas from the construction of $\mathbf{CH}^2_{X/k}$ and some of its key properties [**BW23**, Sections 2 and 3]. First, recall that Grothendieck defines the Picard scheme using the sheafification of the absolute Picard functor. Now consider the codimension 2 Chow group of the threefold $X$. The topological filtration on the Grothendieck group $K_0(X)$ of coherent sheaves on $X$ is defined by setting $F_d(K_0(X))$ to be the subgroup generated by coherent sheaves supported in dimension $\leq d$. Associating an integral curve $C \subset X$ to the class $[\mathcal{O}_C] \in K_0(X)$ induces a morphism $\mathrm{CH}^2(X) \to \mathrm{Gr}^2_F K_0(X) \coloneqq F_1(K_0(X))/F_0(K_0(X))$, which is an isomorphism by Jouanolou's Riemann–Roch theorem without denominators. Motivated by this, Benoist–Wittenberg define the functor $\mathrm{CH}^2_{X/k,\mathrm{fppf}}$ as a certain (presheaf) subquotient of (the fppf sheafification of) a K-theory functor (see [**BW23**, Sections 2.3.2 and 2.2.1]).

Theorem 4.24 (See [**BW23**, Theorem 3.1]). *Let $X$ is a smooth projective geometrically rational threefold over a field $k$. Then $\mathrm{CH}^2_{X/k,\mathrm{fppf}}$ is represented by a smooth group scheme $\mathbf{CH}^2_{X/k}$ over $k$ with the following properties:*

(1) *There is a $G_k$-equivariant isomorphism $\mathrm{CH}^2(X_{\overline{k}}) \to \mathbf{CH}^2_{X/k}(\overline{k})$.*
(2) *The identity component $(\mathbf{CH}^2_{X/k})^0$ is an abelian variety endowed with a principal polarization.*
(3) *Over $\overline{k}$, the identity component $(\mathbf{CH}^2_{X/k})^0$ agrees with Murre's intermediate Jacobian $\mathrm{Ab}^2(X_{\overline{k}})$.*
(4) *The étale group scheme $\mathbf{CH}^2_{X/k}/(\mathbf{CH}^2_{X/k})^0$ is associated to the $G_k$-module $\mathrm{NS}^2(X_{\overline{k}}) \coloneqq \mathbf{CH}^2(X_{\overline{k}})/\,\mathbf{CH}^2(X_{\overline{k}})_{alg}$.*

Thus, we call $(\mathbf{CH}^2_{X/k})^0$ the intermediate Jacobian of $X$. Over non-closed fields, the intermediate Jacobian obstruction (Theorem 4.23) has also been extended [**BW20, HT21b, HT21a, BW23**], and Hassett–Tschinkel and Benoist–Wittenberg furthermore observed that certain *torsors* under the intermediate Jacobian give rise to additional obstructions to rationality. The specific torsors involved are those given by the $k$-points of $\mathbf{CH}^2_{X/k}/(\mathbf{CH}^2_{X/k})^0$, i.e., by the $G_k$-invariant algebraic equivalence classes of 1-cycles on $X_{\overline{k}}$. Note that since the quotient map is a group homomorphism, the $(\mathbf{CH}^2_{X/k})^0$-torsors are additive: for $\gamma, \gamma' \in \mathrm{NS}^2(X_{\overline{k}})^{G_k}$, we have

$$[(\mathbf{CH}^2_{X/k})^{\gamma}] + [(\mathbf{CH}^2_{X/k})^{\gamma'}] = [(\mathbf{CH}^2_{X/k})^{\gamma+\gamma'}]$$

in $H^1(k, (\mathbf{CH}^2_{X/k})^0)$.

Definition 4.25 (Hassett–Tschinkel, Benoist–Wittenberg). Let $X$ is a smooth projective geometrically rational threefold over a field $k$, and assume there exists a smooth projective geometrically connected curve $C$ of genus $\geq 2$ over $k$ such that $(\mathbf{CH}^2_{X/k})^0 \cong \mathbf{Pic}^0_{C/k}$ as principally polarized abelian varieties. We say that the intermediate Jacobian torsor (IJT) obstruction vanishes for $X$ over $k$ if for each $\gamma \in \mathrm{NS}^2(X_{\overline{k}})^{G_k}$, there exists an integer $d$ such that $(\mathbf{CH}^2_{X/k})^{\gamma}$ and $\mathbf{Pic}^d_{C/k}$ are isomorphic as $\mathbf{Pic}^0_{C/k}$-torsors.

Theorem 4.26 (Special case of [**BW23**, Theorems 3.1 and 3.11]). *Let $X$ be a smooth projective geometrically rational threefold over a field $k$.*

(1) *If $X$ is $k$-rational, then there exist a smooth projective (not necessarily connected) curve $D$ over $k$ such that $(\mathbf{CH}^2_{X/k})^0 \cong \mathbf{Pic}^0_{D/k}$ as principally polarized abelian varieties.*
(2) *Assume there exists a smooth projective geometrically connected curve $C$ of genus $\geq 2$ over $k$ such that $(\mathbf{CH}^2_{X/k})^0 \cong \mathbf{Pic}^0_{C/k}$ as principally polarized abelian varieties. If $X$ is $k$-rational, then the IJT obstruction vanishes for $X$ over $k$ (see Definition 4.25).*

As an application, Hassett–Tschinkel and Benoist–Wittenberg characterized $k$-rationality for complete intersections of two quadrics (Theorem 4.2), illustrating the power of the IJT obstruction.

Example 4.27 ([**HT21b, BW23**]). Let $k$ be a field. Let $X = Q_0 \cap Q_1 \subset \mathbb{P}^5_k$ be a smooth complete intersection of quadrics. Recall that $X$ is geometrically rational over $k$ (Example 2.3). The following hold (see [**BW23**, Theorem 4.5]):

- There exists a smooth projective geometrically connected curve $C$ of genus 2 and an isomorphism $(\mathbf{CH}^2_{X/k})^0 \cong \mathbf{Pic}^0_{C/k}$ of principally polarized abelian varieties;
- $\mathrm{NS}^2(X_{\overline{k}})^{G_k} = \mathrm{NS}^2(X_{\overline{k}}) \cong \mathbb{Z}$ is generated by the class $\ell$ of a $\overline{k}$-line on $X_{\overline{k}}$;
- The Hilbert scheme $F_1(X)$ of lines on $X$ is a torsor under $\mathbf{Pic}^0_{C/k}$, and $2[F_1(X)] = [\mathbf{Pic}^1_{C/k}]$ in $H^1(k, \mathbf{Pic}^0_{C/k})$; and
- $F_1(X) \cong (\mathbf{CH}^2_{X/k})^{\ell}$.

We saw in Example 2.3 that if $X$ contains a line over $k$, then $X$ is $k$-rational. Now assume $X$ is $k$-rational. Then, by taking $\gamma = \ell$ in Theorem 4.26, the intermediate Jacobian torsor $(\mathbf{CH}^2_{X/k})^{\ell}$ is isomorphic to $\mathbf{Pic}^d_{C/k}$ for some $d$. Then

$$[\mathbf{Pic}^1_{C/k}] = 2[F_1(X)] = 2[\mathbf{Pic}^d_{C/k}] = [\mathbf{Pic}^{2d}_{C/k}]$$

in $H^1(k, \mathbf{Pic}^0_{C/k})$. Since $C$ has genus 2, the torsor $\mathbf{Pic}^{2d}_{C/k}$ is trivial. This implies that $\mathbf{Pic}^1_{C/k}$ and hence $\mathbf{Pic}^d_{C/k}$ are also the trivial torsor, and hence $\mathbf{Pic}^d_{C/k} \cong (\mathbf{CH}^2_{X/k})^\ell \cong F_1(X)$ has a $k$-point. Therefore $X$ contains a line over $k$.

This example illustrates one possible strategy to leverage the additional data from the IJT obstruction: (1) assume the IJT obstruction vanishes, (2) show that a certain intermediate Jacobian torsor is isomorphic to a certain Hilbert scheme of curves on $X$, (3) use the vanishing of the IJT obstruction to construct a $k$-point on this torsor and hence a curve on $X$ defined over $k$, and (4) use this curve to give a $k$-rationality construction for $X$. In characteristic zero, Kuznetsov–Prokhorov prove that this strategy works for the two other families of Fano threefolds with geometric Picard rank one in Theorem 4.4(2) (see [**KP23**, Theorem 7.2]). This strategy also works for certain quadric surface fibrations coming from pencils of higher-dimensional quadrics:

Example 4.28 ([**JS25**]). Let $k$ be a field of characteristic $\neq 2$. Let $\mathcal{Q} \to \mathbb{P}^1_k$ be a pencil of quadrics in $\mathbb{P}^{2g+1}_k$, and assume that the base locus $V$ of this pencil contains a $(g-2)$-dimensional plane $\Lambda$ defined over $k$. Choose coordinates on $\mathbb{P}^{2g+1}_k$ so that $\Lambda$ is given by $(x_{g-1} = \cdots = x_{2g+1} = 0)$; then $V$ is defined by two quadratic equations of the following form for $i \in \{0, 1\}$

$$\phi_i = l_{i0}x_0 + \cdots + l_{i,g-2}x_{g-2} + q_i,$$

where $l_{ij}, q_i \in k[x_{g-1}, \ldots, x_{2g+1}]$. Now let $X \subset \mathbb{P}^1_k \times \mathbb{P}^{g+2}_k$ be defined by

$$\begin{pmatrix} s & t \end{pmatrix} \begin{pmatrix} l_{00} & \cdots & l_{0,g-2} & q_0 \\ l_{10} & \cdots & l_{1,g-2} & q_1 \end{pmatrix}$$

where $[s : t]$ are coordinates on $\mathbb{P}^1_k$ and $[x_{g-1}, \ldots, x_{2g+1}]$ are coordinates on $\mathbb{P}^{g+2}_k$. Let $C \to \mathbb{P}^1_k$ be the genus $g$ hyperelliptic curve obtained as the Stein factorization of the relative variety of $g$-planes $\mathcal{F}_g(\mathcal{Q}/\mathbb{P}^1)$.

Then the first projection $\psi\colon X \to \mathbb{P}^1_k$ gives $X$ the structure of a quadric surface fibration with $2g + 2$ singular geometric fibers over the branch locus of $C \to \mathbb{P}^1_k$. The following hold (see [**JS25**, Section 4]):

- $(\mathbf{CH}^2_{X/k})^0 \cong \mathbf{Pic}^0_{C/k}$ as principally polarized abelian varieties;
- $\mathrm{NS}^2(X_{\overline{k}})^{G_k} = \mathrm{NS}^2(X_{\overline{k}}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the class $f$ of a $\overline{k}$-fiber of the quadric surface fibration $\psi$ and the class $\sigma$ of a certain $\overline{k}$-section of $\psi$;
- $[\mathbf{Pic}^1_{C/k}] = [(\mathbf{CH}^2_{X/k})^f] = [(\mathbf{CH}^2_{X/k})^{2\sigma}]$ in $H^1(k, \mathbf{Pic}^0_{C/k})$; and
- $F_{g-1}(V) \cong (\mathbf{CH}^2)^{\sigma + af}$ for some $a \in \mathbb{Z}$, and $F_{g-1}(V)$ has a $k$-point if and only if $\psi$ has a section over $k$.

We claim that $X$ is $k$-rational if and only if $\psi$ has a section over $k$. One direction is Example 2.2. For the other direction, if $X$ is $k$-rational, then $[(\mathbf{CH}^2_{X/k})^{\sigma+af}] = [\mathbf{Pic}^d_{C/k}]$ for some $d$ by Theorem 4.26. Note that $\mathbf{Pic}^2_{C/k}(k) \neq \emptyset$ since $C$ is hyperelliptic. This implies $\mathbf{Pic}^{2d}_{C/k}$ and $\mathbf{Pic}^{2a}_{C/k}$ have $k$-points, so the equality of torsors

$$\begin{aligned} [\mathbf{Pic}^{2d}_{C/k}] &= 2[(\mathbf{CH}^2_{X/k})^{\sigma+af}] = [(\mathbf{CH}^2_{X/k})^{2\sigma}] + [(\mathbf{CH}^2_{X/k})^{2af}] \\ &= [(\mathbf{CH}^2_{X/k})^f] + [\mathbf{Pic}^{2a}_{C/k}] = [(\mathbf{CH}^2_{X/k})^f] = [\mathbf{Pic}^1_{C/k}]. \end{aligned}$$

implies $\mathbf{Pic}^1_{C/k}$ has a $k$-point. Then $F_{g-1}(V) \cong (\mathbf{CH}^2)^{\sigma+af} \cong \mathbf{Pic}^d_{C/k}$ has a $k$-point, so $\psi$ has a section.

However, the above strategy does not work in general if $\rho(X_{\overline{k}}) > 1$.

Example 4.29 ([**FJS$^+$24b**]). Let $k$ be a field of characteristic $\neq 2$, and let $X \to \mathbb{P}^1_k \times \mathbb{P}^2_k$ be the double cover branched along a smooth $(2,2)$-divisor, given by

$$w^2 = t_0^2 q_1 + 2t_0 t_1 q_2 + t_1^2 q_3.$$

The second projection $\pi_2 \colon X \to \mathbb{P}^2_k$ is a geometrically standard conic bundle with degree 4 discriminant $\Delta$; assume furthermore that $\Delta$ is smooth. Let $\tilde{\Delta} \to \Delta$ be the discriminant double cover of the conic bundle $\pi_2$, and let $C \to \mathbb{P}^1_k$ be the genus 2 curve defined by $y^2 = -\det(t_0^2 M_1 + 2t_0 t_1 M_2 + t_1^2 M_3)$ where the symmetric $3 \times 3$ matrix $M_i$ corresponds to the quadratic form $q_i$. The following hold (see [**FJS$^+$24b**, Theorems 4.5 and 6.4]):

- $(\mathbf{CH}^2_{X/k})^0 \cong \mathbf{Pic}^0_{C/k}$ as principally polarized abelian varieties;
- $\mathrm{NS}^2(X_{\overline{k}})^{G_k} = \mathrm{NS}^2(X_{\overline{k}}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the class $f$ of a $\overline{k}$-line in a fiber of the quadric surface fibration $\pi_1 \colon X \to \mathbb{P}^1_k$, and the class $\tilde{\gamma}_0$ of a $\overline{k}$-line in a singular fiber of the conic bundle $\pi_2$; and
- $(\mathbf{CH}^2_{X/k})^f \cong \mathbf{Pic}^1_{C/k}$ as $\mathbf{Pic}^0_{C/k}$-torsors.

If the IJT obstruction vanishes for $X$ over $k$, then since $C$ has genus 2, one can check that at least one of $(\mathbf{CH}^2_{X/k})^{\tilde{\gamma}_0}$ or $(\mathbf{CH}^2_{X/k})^{\tilde{\gamma}_0+f}$ must be the trivial torsor.

If, over $k$, there were a(n integral) curve $\sigma$ on $Y$ with algebraic class $\tilde{\gamma}_0$ or $\tilde{\gamma}_0 + f$, then $\sigma$ would give a section of the quadric surface fibration $\pi_1$. However, in general, the existence of a $k$-point on $(\mathbf{CH}^2_{X/k})^{\tilde{\gamma}_0}$ or $(\mathbf{CH}^2_{X/k})^{\tilde{\gamma}_0+f}$ does not imply the existence of such a curve on $X$. Indeed, there exists examples over $\mathbb{R}$ where all the intermediate Jacobian torsors $(\mathbf{CH}^2_{X/k})^{a\tilde{\gamma}_0+bf}$ are trivial but $\pi_1$ has no section over $\mathbb{R}$, e.g., Example 4.15. This is related to Galois descent for codimension 2 cycles, see [**FJS$^+$24b**, Theorem 1.5] and [**FJS$^+$24a**, Section 4.1].

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